BÖTTCHER LAMINATION FOR A COMPLEX POLYNOMIAL

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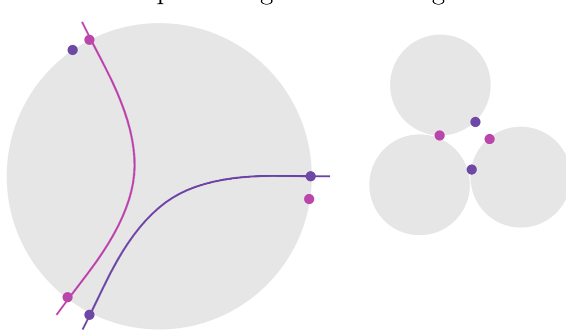
Let f be a polynomial in $\mathbb{C}$ of degree d. We know that f will fix infinity and if $d \geq 2$. the derivative will also vanish at infinity. (This fixed point can then also be called super attractive). We can then write canonical holomorphic coordinates near infinity called Böttcher coordinates, conjugating f to $z \to z^d$. These Böttcher coordinates can be extended along a ray of constant argument from infinity until a precritical point is reached (ie. $p \in \mathbb{C}$ such that $(f^n)(p) = 0$ for some n). The union of the maximal open rays is found in the punctured disk and is made up of half-open segments of the form $l_{v,\theta} = \{z | \arg(z) = \theta, 1 < |z| \leq e^v\}$ for $(v, \theta) \in (\mathbb{R}^+, \mathbb{R}/2\pi\mathbb{Z})$. We call each l a semi-leaf. We will left ∂l denote the endpoint (where some critical or precritical point is hit) of each l, so $\partial l_{v,\theta} = e^{v+i\theta}$.

We can extend the identification with Böttcher coordinates past each precritical point, but then the identification fails to be injective. We can then build a Riemann surface which we call Ω_f that is compatible with the dynamics of f. This Riemann surface, combined with the map from z to d^d is called the Böttcher model for f. Ω_f is created from $\mathbb{C} - D$ by cutting-and-gluing along a backwards-invariant collection of semi-leaves $\{l\}$ which are compatible with the dynamics of f.

The map $z \to z^d$ has $d - 1$ critical points on Ω_f, which correspond to the critical points of f on the complex plane. We then see that there are $(d - 1)$ semi-leaves (counted with multiplicity) that correspond to the critical values of f. To make this Ω_f we then choose two semi-leaves and cut and paste them together to make a leaf. We can indefinitely repeat this process within each chosen full leaf, cutting and pasting the semi-leaves which correspond to precritical values. See the first figure for an example of what this looks like.

Figure 1. Example of the "cutting and pasting" process when $d = 3$. The colored dots on the outside of the disk denote the preimages of a given precritical point. We can then choose any two of these points (where there is no linking) to connect, which we denote here by an arc connecting them in the disk on the left. The disk on the right is a visualization of when we glue these connected points together: dividing the disk into three.



We know that no two critical subsets can have arguments that link in $\mathbb{R}/2\pi\mathbb{Z}$, this means also that no two leaves will cross. This leads to two (related) consequences. First, is that once a cut and paste is chosen, which divides up the disk into two smaller disks, that the preimages of a given critical point in one disk are isolated from the other and any given (valid) choice of a cut and paste must respect this isolation. Second

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— and a result of the first — is that the union of all of these leaves creates an embedding which will be uniquely determined by the placement of critical points. This structure is called a Böttcher lamination of f.

For a more thorough explanation of the correspondence between f and $(\Omega_f, z \to z^d)$, Böttcher coordinates, as well as additional motivation and the text from which the majority of this information was taken, reference D. Calegaris monograph [1]. Notation will be kept as close as possible to the text in the monograph.

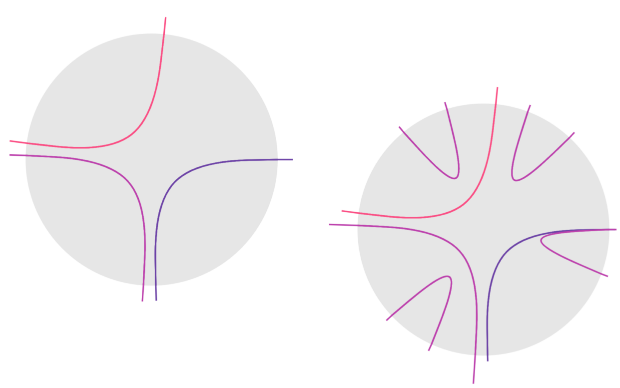
2. Construction, Combinatorics, and Related Matters

For each polynomial of degree d there are $(d-1)$ critical points up to multiplicity and under the preimage of each of these critical points there are mappings of up to d places (again up to multiplicity). For the purposes of ease of construction, we will assume that each critical point is unique and that there are exactly d options for the pullback of each critical point. (If there are less, we can simply skip each duplicate and let there be more options for the subsequent choice — this will be detailed more in later in the text.) We identify each point with an altitude v and an angle θ — so that a given point is found at $e^{v+i\theta}$. We then choose two of these preimages and then cut and paste them together as to unify these points under the preimage. Once we have made this choice for all critical points, we then create d disks that correspond to the possible dynamics of the system: inside each disk there is exactly one (where the pasted together points become one) preimage to choose for each critical point. (Note that this is backwards invariant.)

We can identify each cut and paste by the largest θ (modulo 2π) that is connected, the minimum distance around the circle (in either direction) to reach the preimage which was connected ρ, and the altitude v of the critical point. We will label each of the $(d-1)$ critical points c_i with some ith index where i is chosen such that $v_1 \geq v_2 \geq \cdots \geq v_{d-1}$.

From the no-linking property, not every cut and paste is valid. To decide on whether or not the cut and paste is allowed, the leaves are given priority based on altitude. The leaf with the highest altitude (c_1) can have any choice of cut and paste, while leaves with lower altitudes must choose a cut and paste which does not link with (c_1). The pth precritical leaves for some c_k have length $\frac{v_k}{d^p}$ and hence then become relevant for the construction of the lamination if they have some intermediate altitude for some v_h where $h > k$.

Figure 2. Example of precritical leaves of intermediate altitude: Figure on the left shows three leaves of nearly equal altitudes, so that precritical leaves are not relevant to the construction of the lamination. On the right, the purple leaf has an altitude which is a factor of d greater than the dark blue — and thus the precritical leaves (purple) of the become relevant to the choice of cut and past (for dark blue).

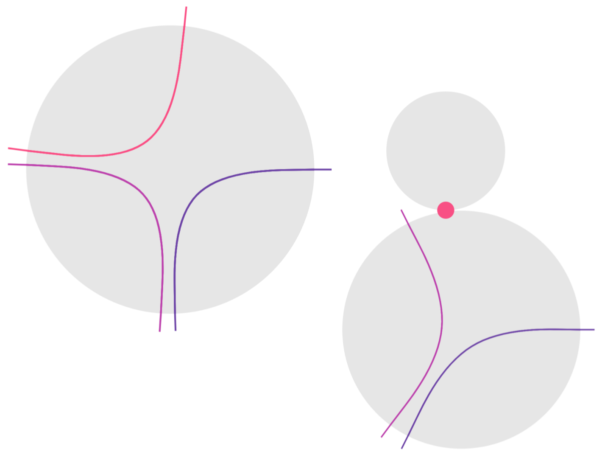


As the lamination of the disk is completely determined once we know the choice of cut and paste for each critical point, we only need to consider all critical leaves and the precritical leaves of intermediate altitude for the construction of the lamination.

We will first consider the case in which no precritical leaves have an intermediate altitude:

[1]D. Calegari, Big Mapping Class Groups and Complex Dynamics

FIGURE 3. Visual of the "reduction" of case when the first cut and paste is made. On the left, is the original lamination. Once we make this cut and paste (done first with the leaf colored pink), further decisions on cut and pastes are isolated on each side of this cut, reducing the lamination to decisions made when $d = 1$ and $d = 2$.



For the first critical leaf (with the highest altitude) there are $(d-2)//2$ ways to split the circle (where $//$ denotes division into integers, and rounding up if d is not even). As we know that the preimages are evenly spaced around the circle and are labeling θ_1 to have the largest argument modulo 2π, we can relabel so that $\theta_1 \in [0, 2\pi(1-1/d))$, adjusting $0 radians$ to be where θ_1 is located. If the spacing has distance greater than $2\pi/d$, then one loses one $1/d$ of the circle as a choice.

Once the cut and paste of c_1 is chosen, we then have a reduced case. When two preimages are glued together, the circle is then divided into two smaller circles. One with circumference ρ_1 and the other with $2\pi - \rho_1$. As there is a no linking property of the leaves, this means that points on either side of this circle are isolated from each other, and that choices that let the two interplay (or link) are invalid perhaps denoting that c_1 in fact did not have the highest altitude or that the choice of cut and paste was wrong.

We can then repeat the procedure for all c_i. Given the reducing quality of the no-linking property, each ensuing cut and paste simplifies the situation to a lower degree. For example. if $d = 6$ and the choice of cut and paste for c_1 divides the circle in half, then the case reduces to $d = 3$ in two circles, where the circumference of the circle is then halved.

We note then that c_{d-1} will have $2\pi/d$ choices for θ_{d-1} from a similar argument for θ_1 when we consider the loss of choices that come from each ensuing paste. In this way, the choice of cut and paste for c_{d-1} is exactly like the choice of cut and paste for $d = 2$, in some way of the circumference of the disk.

Now suppose that some precritical leaf has an intermediate altitude: There will be some k for which $\frac{v_1}{d} \geq v_k \geq v_{k+1} \geq \cdots \geq v_{d-1}$. We know that the critical leaves of c_1 then will take priority for choices of cut and paste over all c_j where $j \geq k$.

As compared to when a precritical leaf did not have an intermediate length, c_j now has less options for the cut and paste and θ_j. The choices for θ_j will be reduced by a multiple of $\frac{1}{d}$, as the critical leaf assumes $\frac{1}{d^2}$ of the circumference of space.

If it is that case that $v_k \leq \frac{v_1}{d^n}$ for some $n > 2$, then the choices for theta will be reduced by a multiple of $(\frac{1}{d^2} + \sum_{i=1}^{n-2} \frac{1}{d^{i+n}} \cdot (d-1)^i)$ ie, each ensuing precritical leaf with a higher altitude will take up space as well, and the number of which will increase by a power of d each time.

3. AMENDMENTS TO A LAMINATION

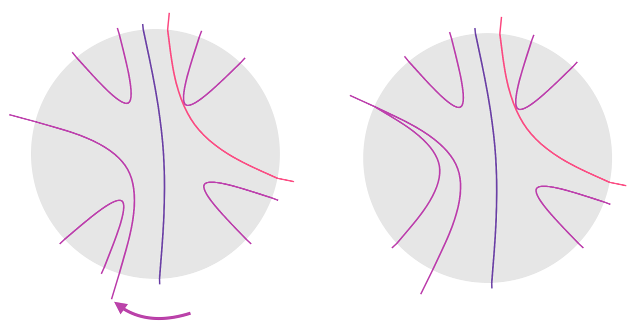
Once a configuration is chosen, it then becomes interesting to consider how this lamination changes if θ_i or some v_i is adjusted. We will write the process for determining the change in a somewhat algorithmic fashion.

(1) If θ_i changes and one leaf bumps into another (ie. causes a link):
   (a) If both leaves are critical (or the same layer of precritical) or the precritical leaf has a greater altitude than the precritical leaf then the leaf (k) with the lower altitude will choose another cut and paste from the available preimages, which does not link with any other leaves of higher altitude.

   If this causes the changed leaf to link with another of a lower altitude, repeat process from part (1) until there are no more links. (Note that as there are a finite amount of leaves, this is a finite process).
   (b) If one leaf is critical and the other is precritical (where the critical leaf altitude is greater than the precritical leaf altitude), the precritical leaf never actually bumps into the critical leaf it instead slides to the other side of the leaf, as it is really traveling along the boundary of the disk that the critical leaf creates when the preimages are merged. (See figure 4.)
(2) If some altitude v_i changes:
   (a) If v_i increased such that c_i has precritical points with intermediate altitude, then choose cut and pastes for all precritical leaves of c_i, where they will not cross any critical leaves with greater altitudes due to the same logic as in ($1.a$).

   If these precritical leaves then link with some critical leaf of a lower altitude, repeat process of ($1.a$).
   (b) If v_i decreases such that c_i ceases to have precritical points with intermediate altitude no immediate changes in the lamination would be made, but ensuing changes of θ_k for any k that previously had a smaller altitude of that precritical point c_i will not have the consider them as a consequence.

Figure 4. Suppose that the dark purple arc has increasing θ and is then moving around the edge of the disk. If it "bumps" into a precritical leaf — as it is about to on the left — then the precritical leaf slides to accommodate the critical leaf (as long as the precritical leaf has a lower altitude).



Note that ($2.b$) implies that the change of θ_i and the changes of v_k for $i, k \in [d]$ are not commutative operations. So a leaf shortening first and then another sliding to another θ could have different results than the reverse if the shortened leaf has precritical leaves of intermediate heights.